Complex Analysis Cheatsheet

A comprehensive cheat sheet covering key concepts, theorems, and formulas in complex analysis, providing a quick reference for students and professionals.

Complex Numbers and Functions

Basic Definitions

Complex Number:	$z = x + iy$, where x and y are real numbers, and i is the imaginary unit ($i^2 = -1$).
Real Part:	$Re(z) = x$
Imaginary Part:	$Im(z) = y$
Complex Conjugate:	$\overline{z} = x - iy$
Modulus:	$\|z\| = \sqrt{x^2 + y^2}$
Argument:	$\theta = \arg(z)$, such that $z = \|z\|e^{i\theta}$

Complex Functions

Definition:	A function $f: \mathbb{C} \rightarrow \mathbb{C}$ that maps complex numbers to complex numbers.
Representation:	$f(z) = u(x, y) + iv(x, y)$, where u and v are real-valued functions.
Limit:	$\lim_{z \to z_0} f(z) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\|f(z) - L\| < \epsilon$ whenever $0 < \|z - z_0\| < \delta$.
Continuity:	$f(z)$ is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.
Derivative:	$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$, if the limit exists.

Polar Form

Representation:	$z = r e^{i\theta} = r(\cos \theta + i \sin \theta)$, where $r = \|z\|$ and $\theta = \arg(z)$
Multiplication:	$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
Division:	$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$
Power:	$z^n = r^n e^{i n \theta}$
Roots:	$z^{1/n} = r^{1/n} e^{i(\theta + 2\pi k)/n}$, for k = 0, 1, ..., n-1

Analytic Functions

Cauchy-Riemann Equations

Equations:	$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
Analyticity:	If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then the Cauchy-Riemann equations hold in D.
Sufficient Condition:	If the partial derivatives of u and v are continuous and satisfy the Cauchy-Riemann equations in a domain D, then f(z) is analytic in D.
Complex Derivative:	$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$
Harmonic Functions:	If f(z) is analytic, then u and v are harmonic functions, i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

Elementary Functions

Exponential Function:	$e^z = e^{x + iy} = e^x(\cos y + i \sin y)$
Trigonometric Functions:	$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$
Hyperbolic Functions:	$\sinh z = \frac{e^z - e^{-z}}{2}$, $\cosh z = \frac{e^z + e^{-z}}{2}$
Logarithmic Function:	$\log z = \ln \|z\| + i \arg z$
Principal Value of Logarithm:	$\text{Log } z = \ln \|z\| + i \text{Arg } z$, where $-\pi < \text{Arg } z \leq \pi$
Complex Power:	$z^c = e^{c \log z}$, where c is a complex constant.

Complex Integration

Contour Integrals

Definition:	$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt$, where C is a smooth curve parameterized by z(t), $a \leq t \leq b$.
Properties:	$\int_C [\alpha f(z) + \beta g(z)] dz = \alpha \int_C f(z) dz + \beta \int_C g(z) dz$ $\int_{-C} f(z) dz = -\int_C f(z) dz$
ML Estimate:	$\|\int_C f(z) dz\| \leq ML$, where M is an upper bound for $\|f(z)\|$ on C, and L is the length of C.

Cauchy's Theorem and Integral Formula

Cauchy's Theorem:	If f(z) is analytic in a simply connected domain D, then for any closed contour C in D, $\oint_C f(z) dz = 0$.
Cauchy's Integral Formula:	If f(z) is analytic in a simply connected domain D, and z_0 is any point in D inside a closed contour C, then $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$.
Generalized Integral Formula:	$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$

Series and Residues

Taylor and Laurent Series

Taylor Series:	If f(z) is analytic in a disk $\|z - z_0\| < R$, then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$.
Laurent Series:	If f(z) is analytic in an annulus $r < \|z - z_0\| < R$, then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, where $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$.

Residue Theorem

Residue:	The residue of $f(z)$ at an isolated singularity z_0 is the coefficient a_{-1} in the Laurent series expansion of $f(z)$ about z_0.
Residue Theorem:	If $f(z)$ is analytic inside and on a closed contour C, except for a finite number of isolated singularities z_k inside C, then $\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$.
Residue Calculation (Simple Pole):	If $f(z)$ has a simple pole at z_0, then $\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$.
Residue Calculation (Pole of Order n):	If $f(z)$ has a pole of order n at z_0, then $\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)]$.

Applications of Residue Theorem

Improper Integrals:	The Residue Theorem can be used to evaluate improper integrals of the form $\int_{-\infty}^{\infty} f(x) dx$.
Trigonometric Integrals:	The Residue Theorem can be used to evaluate integrals of the form $\int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta$.