Real Analysis Cheatsheet

A concise reference for real analysis, covering fundamental concepts, theorems, and techniques. Useful for quick review and problem-solving.

Basic Concepts

Sets and Set Operations

Union (∪)	$A \cup B = \{x : x \in A \text{ or } x \in B\}$
Intersection (∩)	$A \cap B = \{x : x \in A \text{ and } x \in B\}$
Difference ()	$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$
Complement (Ac)	$Ac = \{x : x \in U \text{ and } x \notin A\}$, where U is the universal set.
De Morgan's Laws	$(A \cup B)c = Ac \cap Bc$ $(A \cap B)c = Ac \cup Bc$
Power Set (P(A))	The set of all subsets of A.

Real Numbers and Completeness

Axioms of Real Numbers: Field axioms, order axioms, and the completeness axiom.
Completeness Axiom (Least Upper Bound Property): Every non-empty subset of ℝ that is bounded above has a least upper bound (supremum) in ℝ.
Archimedean Property: For any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.
Density of Rationals: Between any two real numbers, there exists a rational number.
Density of Irrationals: Between any two real numbers, there exists an irrational number.

Sequences

Definition	An ordered list of real numbers: (xn), where $xn \in \mathbb{R}$ for all $n \in \mathbb{N}$.
Convergence	A sequence (xn) converges to x if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\lvert xn - x\rvert < \varepsilon$ for all $n > N$.
Bounded Sequence	There exists $M > 0$ such that $\lvert xn\rvert \leq M$ for all $n \in \mathbb{N}$.
Monotone Sequence	Increasing: $xn \leq xn+1$ for all n. Decreasing: $xn \geq xn+1$ for all n.
Monotone Convergence Theorem	A bounded monotone sequence converges.
Subsequence	A sequence formed from (xn) by selecting some of the elements, usually indexed by a strictly increasing sequence nk.

Limits and Continuity

Limits of Functions

Definition (ε-δ)	$\lim x \to c\ f(x) = L$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < \lvert x - c\rvert < \delta$, then $\lvert f(x) - L\rvert < \varepsilon$.
Sequential Criterion	$\lim x \to c\ f(x) = L$ if and only if for every sequence (xn) converging to c, with $xn \neq c$, the sequence (f(xn)) converges to L.
Limit Laws	Limits of sums, products, quotients (if the denominator's limit is non-zero) follow the expected algebraic rules.
One-Sided Limits	$\lim x \to c+\ f(x)$ (right-hand limit) and $\lim x \to c-\ f(x)$ (left-hand limit).

Continuity

Definition	f is continuous at c if $\lim x \to c\ f(x) = f(c)$.
Sequential Criterion	f is continuous at c if and only if for every sequence (xn) converging to c, the sequence (f(xn)) converges to f(c).
Properties	Sums, products, and compositions of continuous functions are continuous (where defined).
Intermediate Value Theorem (IVT)	If f is continuous on [a, b] and $f(a) \neq f(b)$, then for any value y between f(a) and f(b), there exists $c \in (a, b)$ such that $f(c) = y$.
Extreme Value Theorem (EVT)	If f is continuous on a closed and bounded interval [a, b], then f attains its maximum and minimum values on [a, b].

Uniform Continuity

Definition	f is uniformly continuous on A if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$, if $\lvert x - y\rvert < \delta$, then $\lvert f(x) - f(y)\rvert < \varepsilon$. (δ depends only on ε, not on x).
Heine-Cantor Theorem	If f is continuous on a closed and bounded interval [a, b], then f is uniformly continuous on [a, b].

Differentiation

Definition and Basic Theorems

Derivative Definition	$f'(x) = \lim h \to 0\ (f(x + h) - f(x)) / h$, if the limit exists.
Differentiability Implies Continuity	If f is differentiable at x, then f is continuous at x.
Rules of Differentiation	Sum, product, quotient, and chain rules.

Mean Value Theorems

Rolle's Theorem: If f is continuous on [a, b], differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.
Mean Value Theorem (MVT): If f is continuous on [a, b] and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = (f(b) - f(a)) / (b - a)$.
Consequences of MVT: If $f'(x) = 0$ for all x in an interval, then f is constant on that interval. If $f'(x) > 0$ (or $f'(x) < 0$) on an interval, then f is increasing (or decreasing) on that interval.

L'Hôpital's Rule

Indeterminate Forms	0/0, ∞/∞, 0 * ∞, ∞ - ∞, 1^∞, 0^0, ∞^0
L'Hôpital's Rule	If $\lim x \to c\ f(x) = 0$ and $\lim x \to c\ g(x) = 0$ (or both are ∞) and $\lim x \to c\ f'(x)/g'(x)$ exists, then $\lim x \to c\ f(x)/g(x) = \lim x \to c\ f'(x)/g'(x)$.

Integration

Riemann Integration

Partition	A partition P of [a, b] is a finite set of points $\{x_0, x_1, ..., x_n\}$ such that $a = x_0 < x_1 < ... < x_n = b$.
Upper and Lower Sums	$U(f, P) = \Sigma M_i(x_i - x_{i-1})$, where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ $L(f, P) = \Sigma m_i(x_i - x_{i-1})$, where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$
Riemann Integral	f is Riemann integrable on [a, b] if the upper and lower integrals are equal. The common value is the Riemann integral $\int_a^b f(x)\,dx$.
Integrability Condition	f is Riemann integrable if and only if for every $\varepsilon > 0$, there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$.

Fundamental Theorem of Calculus

FTC Part 1:
If f is continuous on [a, b] and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ for $x \in (a, b)$.

FTC Part 2:
If f is continuous on [a, b] and F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Improper Integrals

Type 1 (Infinite Interval)	$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$
Type 2 (Discontinuous Integrand)	If f is discontinuous at $c \in (a, b)$, then $\int_a^b f(x)\,dx = \lim_{t\to c^-} \int_a^t f(x)\,dx + \lim_{t\to c^+} \int_t^b f(x)\,dx$