Differential Equations Cheatsheet

A comprehensive cheat sheet covering essential concepts, formulas, and methods for solving differential equations, including first-order, second-order, and higher-order equations.

First-Order Differential Equations

Basic Forms and Definitions

A **differential equation** is an equation involving derivatives of a function.

A **first-order differential equation** involves only the first derivative.

General form: `dy/dx = f(x, y)`

An **explicit solution** is a function `y = φ(x)` that satisfies the differential equation.

A **general solution** contains arbitrary constants.

An **implicit solution** is a relation `G(x, y) = 0` that defines a solution implicitly.

An **initial value problem** (IVP) consists of a differential equation and an initial condition `y(x_0) = y_0`.

Separable Equations

Form	`dy/dx = f(x)g(y)`
Solution	`∫ dy/g(y) = ∫ f(x) dx`
Example	`dy/dx = x/y` => `∫ y dy = ∫ x dx` => `y^2/2 = x^2/2 + C`

Linear Equations

Form	`dy/dx + P(x)y = Q(x)`
Integrating Factor	`μ(x) = e^(∫ P(x) dx)`
Solution	`y(x) = (1/μ(x)) ∫ μ(x)Q(x) dx`
Example	`dy/dx + y = x` => `μ(x) = e^x` => `y(x) = e^(-x) ∫ e^x x dx = x - 1 + Ce^(-x)`

Exact Equations

Form	`M(x, y) dx + N(x, y) dy = 0`
Test for Exactness	`∂M/∂y = ∂N/∂x`
Solution	`∫ M(x, y) dx + ∫ [N(x, y) - ∂/∂y ∫ M(x, y) dx] dy = C`
Example	`(2x + y)dx + (x + 3y^2)dy = 0` is exact. Solution: `x^2 + xy + y^3 = C`

Homogeneous Equations

Form	`dy/dx = f(x, y)`, where `f(tx, ty) = f(x, y)` for all `t`.
Substitution	`v = y/x` or `y = vx`, then `dy/dx = v + x(dv/dx)`
Example	`dy/dx = (x^2 + y^2) / (xy)`. Let `y = vx`. Resulting separable equation can be solved.

Bernoulli Equations

Form	`dy/dx + P(x)y = Q(x)y^n`
Substitution	`v = y^(1-n)`
Transformed Equation	`dv/dx + (1-n)P(x)v = (1-n)Q(x)` (linear in `v`)

Second-Order Linear Homogeneous Equations

General Form

`ay'' + by' + cy = 0`, where `a`, `b`, and `c` are constants.

The **characteristic equation** is `ar^2 + br + c = 0`.

The roots `r_1` and `r_2` determine the form of the general solution.

Distinct Real Roots ($r_1 \neq r_2$)

General Solution	`y(x) = c_1e^(r_1x) + c_2e^(r_2x)`
Example	For `y'' - 3y' + 2y = 0`, `r_1 = 1`, `r_2 = 2`. So, `y(x) = c_1e^x + c_2e^(2x)`

Repeated Real Roots ($r_1 = r_2 = r$)

General Solution	`y(x) = c_1e^(rx) + c_2xe^(rx)`
Example	For `y'' - 4y' + 4y = 0`, `r = 2`. So, `y(x) = c_1e^(2x) + c_2xe^(2x)`

Complex Conjugate Roots ($r = \alpha \pm \beta i$)

General Solution	`y(x) = e^(αx)(c_1cos(βx) + c_2sin(βx))`
Example	For `y'' + 2y' + 5y = 0`, `r = -1 ± 2i`. So, `y(x) = e^(-x)(c_1cos(2x) + c_2sin(2x))`

Initial Value Problems

Given `y(x_0) = y_0` and `y'(x_0) = y_1`, solve for `c_1` and `c_2` using the initial conditions.

Substitute `x_0` into the general solution and its derivative, then solve the resulting system of equations.

Second-Order Linear Non-Homogeneous Equations

General Form

`ay'' + by' + cy = g(x)`, where `a`, `b`, and `c` are constants and `g(x) ≠ 0`.

The general solution is `y(x) = y_c(x) + y_p(x)`, where `y_c(x)` is the complementary solution and `y_p(x)` is a particular solution.

`y_c(x)` is the general solution to the homogeneous equation `ay'' + by' + cy = 0`.

Method of Undetermined Coefficients

Applicable when	`g(x)` is a polynomial, exponential, sine, cosine, or a combination of these.
Procedure	Assume a form for `y_p(x)` based on `g(x)`, with undetermined coefficients. Substitute into the differential equation to find the coefficients.
Example (Polynomial)	If `g(x) = x^2`, assume `y_p(x) = Ax^2 + Bx + C`
Example (Exponential)	If `g(x) = e^(kx)`, assume `y_p(x) = Ae^(kx)`
Example (Sine/Cosine)	If `g(x) = sin(kx)`, assume `y_p(x) = Acos(kx) + Bsin(kx)`

Variation of Parameters

Formula	`y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)`
Where	`u_1'(x) = -y_2(x)g(x) / W(y_1, y_2)` `u_2'(x) = y_1(x)g(x) / W(y_1, y_2)`
Wronskian	`W(y_1, y_2) = y_1y_2' - y_2y_1'`
General Solution	`y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)`

Laplace Transforms

Definition

The Laplace Transform of a function `f(t)` is defined as:

`F(s) = L{f(t)} = ∫₀^∞ e^(-st) f(t) dt`

Where `s` is a complex number and the integral converges.

Basic Laplace Transforms

`L{1}`	`1/s`, `s > 0`
`L{t^n}`	`n! / s^(n+1)`, `s > 0`, `n` is a non-negative integer
`L{e^(at)}`	`1 / (s - a)`, `s > a`
`L{sin(at)}`	`a / (s^2 + a^2)`, `s > 0`
`L{cos(at)}`	`s / (s^2 + a^2)`, `s > 0`

Properties of Laplace Transforms

Linearity	`L{af(t) + bg(t)} = aL{f(t)} + bL{g(t)}`
Derivative	`L{f'(t)} = sF(s) - f(0)`
Second Derivative	`L{f''(t)} = s^2F(s) - sf(0) - f'(0)`
Translation in s	`L{e^(at)f(t)} = F(s - a)`
Translation in t	`L{f(t - a)u(t - a)} = e^(-as)F(s)`, where `u(t)` is the Heaviside step function
Convolution	`L{(f * g)(t)} = F(s)G(s)`

Solving Differential Equations with Laplace Transforms

1. Take the Laplace transform of both sides of the differential equation.
2. Use initial conditions and properties of Laplace transforms to express the equation in terms of `F(s)`.
3. Solve for `F(s)`.
4. Take the inverse Laplace transform of `F(s)` to find `f(t)`.