Linear Algebra Cheatsheet

A concise reference for key concepts, formulas, and operations in linear algebra. This cheat sheet covers vectors, matrices, linear transformations, and more, providing a quick guide for students, engineers, and researchers.

Vectors and Spaces

Basic Vector Operations

Vector Addition	\mathbf{u} + mathbf{v} = (u_1 + v_1, u_2 + v_2, ..., u_n + v_n)
Scalar Multiplication	cmathbf{u} = (cu_1, cu_2, ..., cu_n)
Dot Product	\mathbf{u} cdot mathbf{v} = u_1v_1 + u_2v_2 + ... + u_nv_n
Vector Norm (Magnitude)	\lVert mathbf{u} Vert = sqrt{u_1^2 + u_2^2 + ... + u_n^2}
Cross Product (3D)	\mathbf{u} imes mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)
Unit Vector	\hat{mathbf{u}} = \frac{mathbf{u}}{\lVert mathbf{u} Vert}

Vector Spaces

Vector Space Axioms:
A set V is a vector space over a field F if it satisfies the following axioms for all \mathbf{u}, \mathbf{v}, \mathbf{w} \in V and c, d \in F:

1. \mathbf{u} + \mathbf{v} \in V (Closure under addition)
2. \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} (Commutativity of addition)
3. (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) (Associativity of addition)
4. There exists a zero vector \mathbf{0} \in V such that \mathbf{u} + \mathbf{0} = \mathbf{u} (Existence of additive identity)
5. For each \mathbf{u} \in V, there exists -\mathbf{u} \in V such that \mathbf{u} + (-\mathbf{u}) = \mathbf{0} (Existence of additive inverse)
6. c\mathbf{u} \in V (Closure under scalar multiplication)
7. c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} (Distributivity of scalar multiplication over vector addition)
8. (c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u} (Distributivity of scalar multiplication over field addition)
9. c(d\mathbf{u}) = (cd)\mathbf{u} (Compatibility of scalar multiplication with field multiplication)
10. 1\mathbf{u} = \mathbf{u} (Identity element of scalar multiplication)

Subspaces

Definition	A subset W of a vector space V is a subspace if it is itself a vector space under the same operations defined on V.
Conditions for a Subspace	To prove W is a subspace of V, show: 1. W is non-empty (i.e., \mathbf{0} \in W). 2. W is closed under addition: If \mathbf{u}, \mathbf{v} \in W, then \mathbf{u} + \mathbf{v} \in W. 3. W is closed under scalar multiplication: If \mathbf{u} \in W and c is a scalar, then c\mathbf{u} \in W.
Examples	• The set containing only the zero vector, {\mathbf{0}}, is a subspace. • The entire vector space V is a subspace of itself. • A line through the origin in \mathbb{R}^2 is a subspace of \mathbb{R}^2.

Matrices

Basic Matrix Operations

Matrix Addition	(A + B)_{ij} = A_{ij} + B_{ij} (element-wise addition)
Scalar Multiplication	(cA)_{ij} = c(A_{ij}) (multiply each element by the scalar)
Matrix Multiplication	(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} (row i of A times column j of B)
Transpose	(A^T)_{ij} = A_{ji} (swap rows and columns)
Trace	\text{tr}(A) = \sum_{i=1}^{n} A_{ii} (sum of diagonal elements)
Determinant	\det(A) (a scalar value that can be computed recursively or by row reduction)
Inverse	A^{-1} (a matrix such that AA^{-1} = A^{-1}A = I, where I is the identity matrix)

Special Matrices

- **Identity Matrix (I)**: A square matrix with 1s on the main diagonal and 0s elsewhere.
- **Zero Matrix**: A matrix with all elements equal to 0.
- **Diagonal Matrix**: A square matrix with non-zero elements only on the main diagonal.
- **Symmetric Matrix**: A square matrix A such that A = A^T.
- **Skew-Symmetric Matrix**: A square matrix A such that A = -A^T.
- **Orthogonal Matrix**: A square matrix Q such that Q^TQ = QQ^T = I.

Matrix Properties

Associativity	(AB)C = A(BC)
Distributivity	A(B + C) = AB + AC (A + B)C = AC + BC
Scalar Multiplication	c(AB) = (cA)B = A(cB)
Transpose Properties	(A + B)^T = A^T + B^T (cA)^T = cA^T (AB)^T = B^TA^T
Inverse Properties	(A^{-1})^{-1} = A (AB)^{-1} = B^{-1}A^{-1}
Determinant Properties	\det(AB) = \det(A)\det(B) \det(A^T) = \det(A) \det(A^{-1}) = \frac{1}{\det(A)}

Linear Transformations

Definition and Properties

Definition	A linear transformation $T: V \to W$ is a function between vector spaces V and W that preserves vector addition and scalar multiplication.
Properties	1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$. 2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in V$ and scalar c.
Zero Vector	$T(\mathbf{0}_V) = \mathbf{0}_W$, where $\mathbf{0}_V$ and $\mathbf{0}_W$ are the zero vectors in V and W, respectively.
Linear Combination	$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \dots + c_nT(\mathbf{v}_n)$

Kernel and Image

Kernel (Null Space)	$\text{ker}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}_W\}$. The kernel is a subspace of V.
Image (Range)	$\text{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$. The image is a subspace of W.
Rank-Nullity Theorem	$\dim(\text{ker}(T)) + \dim(\text{im}(T)) = \dim(V)$

Matrix Representation

Given a linear transformation $T: V \to W$, and bases $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for V and $C = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ for W, the matrix representation of T with respect to B and C is the $m \times n$ matrix A such that:

$[T(\mathbf{v})]_C = A[\mathbf{v}]_B$

where $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_C$ are the coordinate vectors of $\mathbf{v}$ and $T(\mathbf{v})$ with respect to the bases B and C, respectively.

The columns of A are the coordinate vectors of $T(\mathbf{v}_i)$ with respect to the basis C, i.e., $A = [[T(\mathbf{v}_1)]_C \ \ [T(\mathbf{v}_2)]_C \ \ \dots \ \ [T(\mathbf{v}_n)]_C]$

Eigenvalues and Eigenvectors

Definitions

Eigenvalue	A scalar λ is an eigenvalue of a square matrix A if there exists a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
Eigenvector	A non-zero vector $\mathbf{v}$ is an eigenvector of a square matrix A corresponding to the eigenvalue λ if $A\mathbf{v} = \lambda\mathbf{v}$.
Eigenspace	The eigenspace of A corresponding to the eigenvalue λ is the set of all eigenvectors corresponding to λ, together with the zero vector. It is a subspace of $\mathbb{R}^n$ and is denoted by $E_\lambda = \{\mathbf{v} : A\mathbf{v} = \lambda\mathbf{v}\}$.

Finding Eigenvalues and Eigenvectors

To find the eigenvalues of a matrix A, solve the characteristic equation:

$\det(A - \lambda I) = 0$

where I is the identity matrix and λ is the eigenvalue.

Once the eigenvalues are found, the corresponding eigenvectors can be found by solving the equation:

$(A - \lambda I)\mathbf{v} = \mathbf{0}$

for each eigenvalue λ.

Diagonalization

Diagonalizable Matrix	A square matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. The columns of P are the eigenvectors of A, and the diagonal entries of D are the corresponding eigenvalues.
Conditions for Diagonalization	An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.
Procedure	1. Find n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ of A. 2. Form the matrix P whose columns are these eigenvectors: $P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ \dots \ \mathbf{v}_n]$. 3. Form the diagonal matrix D with the corresponding eigenvalues on the diagonal: $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. 4. Then $A = PDP^{-1}$.